AAL, HW1 solution

1. Let H and K be subgroups of the group G. Prove that HK is a subgroup of G if and only if $HK = KH$.

   Solution:

   Assume first that $HK = KH$.
   $1 = 1 * 1 \in HK$ since 1 is contained in every subgroup of G.
   We prove that HK is closed under multiplication: Let $h_1, h_2 \in H$, $k_1, k_2 \in H$. Then $(h_1k_1)(h_2k_2) = h_1(k_1h_2)k_2$ can be written as $h_1(h_3k_3)k_2$ for some $h_3 \in H$, $k_3 \in K$ since $HK = KH$. Now $h_1(h_3k_3)k_2 = (h_1h_3)(k_3k_2) \in HK$ since $h_1h_3 \in H$ and $k_3k_2 \in K$ follows from the fact that H and K are subgroups.

   Assume now that HK is a subgroup.

   Let $kh \in KH$ ($h \in H$, $k \in K$). Clearly, $(kh) = (h^{-1}k^{-1})^{-1} \in HK$ since HK is a subgroup of G. Thus $KH \subseteq HK$.

   Let $hk \in H$ ($h \in H$, $k \in K$). Then $(hk)^{-1} \in HK$ since $HK \leq G$ so $(hk)^{-1} = h_1k_1$ for some $h_1 \in H$, $k_1 \in K$. This implies $hk = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH$ since K and H are inverse closed.

2. Let G be the set of upper triangular $3 * 3$ matrices over the field $\mathbb{F}_3$, whose diagonal elements are 1.

$$G := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_3 \right\}$$

   G is a group with respect to matrix multiplications show that

   (a) Verify that every nonidentity element of G is of order 3.
   Solution:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 3a & 3c + 3ab \\ 0 & 1 & 3b \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

   Then the order of this element divides 3 so it can only be 1 since the only element of order 1 in a group is the identity element.

(b) Calculate the center of the group G.

Solution:

$$\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+x & ay+c+z\\ 0 & 1 & b+y\\ 0 & 0 & 1\end{pmatrix},$$

$$\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+x & bx+c+z\\ 0 & 1 & b+y\\ 0 & 0 & 1\end{pmatrix}.$$

It follows that if $\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}$ is in the center of G, then $ay = bx$ for every $x, y \in \mathbb{F}_3$. If $a \neq 0$, then $ay \neq bx$ if $x = 0$ and $y = 1$, while if $b \neq 0$, then $ay \neq bx$ if $y = 0$ and $x = 1$. Thus $a = 0$ and $b = 0$.

On the other hand G is a p-group so its center is nontrivial so it is or order at least 3. Therefore $Z(G) = \left\{\begin{pmatrix}1 & 0 & c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \mid c \in \mathbb{F}_3\right\}$.

Simple matrix calculations show that

3. Assume that $G/\mathrm{Z}(G)$ is cyclic. Prove that G is abelian.

   Solution: Every element of can be written as $x^i z$, where $xZ(G)$ generates $G/Z(G)$ and $z \in Z(G)$. Then it is easy to see that $x^i z$ and $x^j z'$ commute since z and z' are in $Z(G)$.